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are discussed in the classical way and Richmond's construction for the polygon of 17 sides is given as an example. In the third chapter the question of constructions to be carried out by ruler alone is developed in detail with the separation of projective and metrical properties. Mathew's *Projective Geometry* is drawn upon extensively for the discussion of homography and its special case of involution to establish the constructions. The relation of the infinity locus to metrical constructions is unusually well presented.

In Chapter IV the fundamental theorem concerning the possibility of solution of a quadratic equation with the aid of ruler and compasses is established, two solutions being given, and, from the discussion of a pencil of conics projected from a circle through four points, a construction for the common points of projective point rows superposed on a line is obtained. A second division of this chapter treats of modern instruments, including dividers, Hilbert's Einheitsdreher, the parallel ruler, and the set-square. In Chapter V standard methods of attack for various problems are discussed. No pretense is made at developing a general method for all problems, but the data in any given case are used to point toward one of seven listed methods of solution. This is in some respects the best organized chapter of the book. In Chapter VI the methods just indicated are compared, especially as regards constructions to be accomplished in a limited space or with a minimum number of operations. The two concluding chapters are concerned with effecting all ruler and compass constructions with one fixed circle and ruler only, or with compasses only. They may be regarded as addenda to the preceding discussion, inasmuch as they are not necessary to its development. In the chapter on compasses only, the results of Mascheroni and Adler have been compared and several solutions by the methods of each given as examples.

On the whole, the teacher of analytic or projective geometry or of college algebra will find this a valuable source-book for many illustrative problems. Its development of fundamental theorems of projective geometry for the general conic starting from properties of the circle is carried out in an interesting way. The sources of many theorems are carefully indicated and a fairly complete bibliography is given facing page 1. In the way of criticism, the reviewer believes that some of the section headings are misplaced and that many of them are not illuminating. The chapter on methods of solution is an exception to this criticism.

B. M. WOODS.

THE UNIVERSITY OF CALIFORNIA.

A Treatise on the Circle and the Sphere. By JULIAN LOWELL COOLIDGE, Assistant Professor of Mathematics in Harvard University. The Clarendon Press, Oxford, 1916. 604 pages. $6.75.

Professor Coolidge has written a treatise on the circle and the sphere which may be considered as an encyclopedia of valuable information on this important subject. The successful compilation of known material of such an extent, and the incorporation of the results of his own notable investigations in this field, in a uniform mold, could only be accomplished by a great amount of work and a full mastery of the subject by the author. The contents are too voluminous to be discussed in detail in a review of moderate size and scope. I shall therefore restrict myself, in the main, to some of the important features of the book and of the subject.

The first chapter deals with the elementary geometry of the circle, in which inversion, mutually tangent circles, circles related to a triangle, the Brocard figures, concurrent and coaxial circles, are considered. Out of the almost innumerable mass of propositions of the so-called modern geometry of the triangle and the circle, a great number of theorems have been selected and proved by the usual well-known means.

This subject is continued in a second chapter by making use of cartesian and trilinear coördinates and applying them to the nine-point circle, the Tucker circles, the Brocard circles, etc.

Next, an important step forward is made by the introduction of tetracyclic coördinates. Choosing any four mutually orthogonal circles not null as fixed, and any point P in the plane of these circles, the tetracyclic coördinates x_1, x_2, x_3, x_4 of P are defined as quantities proportional to the ratios of the powers of P to the radii of the corresponding circles. The coördinates of any point P satisfy the identity

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

are homogeneous, and do not all vanish simultaneously. The symbol (yx) stands for

$$(yx) = y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4,$$

and $(yx) = 0$ is the equation of a circle. The coefficients (y) are called the coördinates of the circle. If they satisfy the foregoing identity, the point (y) is called the vertex of the circle, which is then said to be null. The angle θ between two proper circles with the coördinates (y) and (z) is defined by

$$\cos \theta = \frac{(yz)}{\sqrt{yy}\sqrt{zz}}.$$

The conditions for orthogonality or tangency of the two circles are clearly $(yz) = 0$, and $(yy)(zz) - (yz)^2 = 0$. A homogeneous algebraic equation, $f(x_1, x_2, x_3, x_4) = 0$, in conjunction with the given identity, defines a cyclic curve. Thus, when f is a polynomial of the second degree, it represents, in general, a bicircular quartic. It goes without saying that the important relation between the tetracyclic plane and three-space is fully explained. For example, it is shown how a one-to-one correspondence may be established between the circles of the cartesian plane and the points of cartesian space.

"The coördinates of a circle in the cartesian plane may be interpreted as the coördinates of a point in space whose polar plane with regard to a fundamental sphere cuts that sphere in a circle whose stereographic projection is the given circle."

Again: "The homogeneous coördinates (x) may be taken to represent a point in a three-dimensional space, which we shall assume has an elliptic type of measurement, the equation of the absolute quadric being

$$(xx) = 0."$$

In Chapter VI, quite in analogy with the corresponding development in the tetracyclic plane, pentaspherical coördinates and space are introduced and applied to the investigation of systems of spheres.[1] In both, the tetracyclic plane and the pentaspherical space, much use is made of the identity of Darboux and Frobenius between the coördinates of any ten circles, or any twelve spheres, which plays a fundamental rôle in the theory.

At this point I should like to say that for one familiar with the essentials of tetracyclic and pentaspheric coördinates there is, of course, no difficulty in following the author's exposition of these systems. The beginner, however, we fear, will not have very smooth sailing, if he has not the help of a teacher, or some other source for consultation. The style of Darboux's own presentation in Chapter VI, Vol. 1, of his *Leçons sur la Théorie Générale des Surfaces*, has not been surpassed, and also Bôcher's more explicit account in his *Reihenentwicklungen der Potentialtheorie* might have been successfully used as a model of clear exposition. Reference should also be made to the fact that it was Darboux who first introduced pentaspherical coördinates[2] for the investigation of certain curves and surfaces.

In the following chapters, cyclides, circle-transformations, continuous groups of transformations, sphere transformations, Laguerre transformations, oriented circles and spheres, circles orthogonal to one sphere, circle-crosses, algebraic systems, and differential geometry of circle systems are discussed. A number of topics in this series contain some of the author's own contributions to the geometry of circles and spheres. Oriented circles and spheres, based upon positive and negative radii, and the direction of the normals make it possible to define uniquely the angle included by two circles, or two spheres, or a circle and a sphere. From a purely constructive standpoint Chapter III, dealing with famous problems, is of particular interest, and it was a happy thought of the author to pay special attention to these problems. The main portions of this chapter are presented in an admirable manner. It opens with Lemoine's geometrographic criteria to which possibly too much prominence is given. They are hardly of any practical value, insofar as they do not indicate how to simplify a construction or how to make it more accurate.[3]

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[1] The foundation for a general theory of spheres in n-space was laid by G. Schlumberger in his dissertation: *Über n-dimensionale lineare und quadratische Kugelsysteme* (78 pages), Zürich, 1896.

[2] *Sur une class remarquable de courbes et de surfaces*, p. 135. Paris, 1873. See also reference given above.

[3] See in this connection Konrad Nitz: "Beiträge zu einer Fehlertheorie der geometrischen Konstruktionen," *Zeitschrift für Mathematik und Physik*, Vol. 53, pp. 1–37 (1906); also Em. Haentschel's account in *L'Enseignement Mathématique*, Vol. 9, pp. 45–51 (1907).

The author makes some remarks on the proofs of Steiner's solution of Malfattis' problem: "to construct three circles, each of which shall touch the other two, and two sides of a given triangle," which are misleading. Coolidge says with reference to Hart's proof: "There is a suspicion which naturally arises that, if the first discoverer of a proof had been of Steiner's own nationality, less trouble would have been given to disparaging his work." I doubt that there was any national feeling which induced criticism of Hart's proof. In the first place, Steiner was a Swiss. In the second place, Steiner, the discoverer of inversion, undoubtedly used this principle in the solution of this and similar problems. Criticism was due solely to the demand that, in the proofs, the principles discovered and used by Steiner, which form the natural foundation for a discussion of these problems, should be followed. In fact, Plücker, a German, who was not able to understand and appreciate Steiner's powerful methods, made the insulting remark, in print, that Steiner presumably had no proof at all for his construction. As indicated above, Schröter's remarks on Hart's proof were, therefore, certainly dictated by purely scientific motives only.

Malfattis's problem has an equally famous counterpart in Apollonius's problem: "To construct a circle tangent to three given circles." Also the solution of this problem caused many controversies. Gergonne, whose solution is given, claimed that analytic methods were superior to synthetic methods, whereupon Poncelet[1] published a purely geometric solution. No reference is given to Poncelet in this connection. Fiedler's cyclographic method of solving certain circle-problems is fully discussed. Here we see again, for example in the solution of Apollonius's problem, that for certain classes of problems there are certain "natural" methods leading to their solutions.

The general make-up and the typography of the book are excellent. In a number of places the author makes rather abrupt statements; for example, on page 130: "We distinguish the following types of circles:

$$\sum_{i=0}^{i=3} y_i z_i = (yz).$$

(*a*) Proper circles $(xx) \neq 0$, $ix_0 + x_1 \neq 0$, etc."

Considered all in all, Coolidge's treatise is a great and most up to date work of reference and important information on the theory of the circle and the sphere. It is indispensable for anyone who intends to take up this field as a student, or as an investigator.

ARNOLD EMCH.

UNIVERSITY OF ILLINOIS.

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[1] *Traite des propriétés projéctives des figures*, Vol. 1, pp. 136–148, Paris, 1865. Also *Applications d'analyse et de géométrie*, Vol. 1, pp. 30–41 (1862).